http://www.wholikeshomework.com

Partial Fractions and Integration

In this lesson we are going to review the basics of integration with partial fractions. Partial fractions are used for the integration of *rational functions*, or functions which are a ratio of polynomials. In other words, for functions of the form

$f(x) = \dfrac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are both polynomials.

Examples of such functions include

$$\frac{x^2 - 3x + 5}{x^2 - 4} \qquad \frac{x + 3x^2}{x^3 + 2x} \qquad \frac{2x - 6}{x^3 - 6x^2 + 5x - 30}$$

Traditionally, such functions prove to be either difficult or impossible to integrate. However, the goal of partial fractions is to break these functions apart into sums of fractions that are far easier to integrate. To illustrate the general idea, take for example the first function listed.

$$\frac{3x + 8}{x^2 - 4}$$

We can break this apart into a sum of two fractions such that

$$\frac{3x + 8}{x^2 - 4} = \frac{-0.5}{x + 2} + \frac{3.5}{x - 2}$$

To prove this, add the fractions together by multiplying each fraction by the common denominator. In this case, it is $(x + 2)(x - 2)$

$$\begin{aligned}
\frac{3x + 8}{x^2 - 4} &= \frac{-0.5}{x + 2} + \frac{3.5}{x - 2} \\
&= \frac{-0.5}{x + 2} \cdot \frac{(x - 2)}{(x - 2)} + \frac{3.5}{x - 2} \cdot \frac{(x + 2)}{(x + 2)} \\
&= \frac{-0.5x + 1}{(x + 2)(x - 2)} + \frac{3.5x + 7}{(x + 2)(x - 2)} \\
&= \frac{-0.5x + 1 + 3.5x + 7}{(x + 2)(x - 2)} \\
&= \frac{3x + 8}{x^2 - 4}
\end{aligned}$$

Copyright © 2007 Who Likes Homework - All rights reserved.

Evaluating $\int \frac{3x+8}{x^2-4}dx$, which would ordinarily prove difficult, becomes $\int \frac{-0.5}{x+2}+\frac{3.5}{x-2}dx$, which is easily computed ($-0.5\ln(x+2)+3.5\ln(x-2)+c$).

To transform and integrate these rational functions takes a tedious and stepwise process. Learning may take a little time, but with some practice the routine should become relatively easy. To illustrate the process, we will walkthrough $\int \frac{2x-6}{x^3-6x^2+5x-30}dx$

Step 1 – Factor out the denominator as much as possible

- The example above has the easiest situation possible, where all of the factors turn out to be linear terms. Another example would be
  $$\frac{x+1}{x^3+2x^2-3x}=\frac{x+1}{x(x+3)(x-1)}$$
- However, functions may not always decompose into linear terms. Try to get the lowest order terms as possible. For the example we are walking through,
  $$\frac{2x-6}{x^3-6x^2+5x-30}=\frac{2x-6}{(x^2+5)(x-6)}$$

  Notice in this example, the denominator is factored as far as possible, as x^2+5 cannot be factored any further. However, this is still useful for what we are trying to accomplish. Higher order (> 2) can be done, but for the purposes of this lesson only linear and quadratic factors will be considered.

Step 2 –Turn the function to a sum of fractions, whose denominators correspond to the factors determined in Step 1. The numerator will be an expression with unknown constants.

- In other words, for every factor that you were able to determine, create a fraction whose denominator *is the factor* and the numerator is unknown.
  - For factors that are linear (ax + b), the form of the fraction will be $\frac{A}{ax+b}$, where A, a, and b are constants.
  - For factors denominators that are quadratic (ax2 + bx + c), the form of the fraction will be $\frac{Ax+B}{ax^2+bx+c}$, where A, B, a, b and c are constants.
- Each of these factors will be summed together. For example, if the function we factored is $\frac{x+1}{x(x+3)(x-1)}$, we will then have
  $$\frac{x+1}{x(x+3)(x-1)}=\frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-1}$$

Copyright © 2007 Who Likes Homework - All rights reserved.

Likewise, to illustrate quadratic factors, if we have $\frac{x-4}{(x^2+4)(x^2+6)}$

$$\frac{x-4}{(x^2+4)(x^2+6)}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+6}$$

- Both linear and quadratic fractions can be summed together without problems. For the walkthrough example, we will have
$$\frac{2x-6}{(x^2+5)(x-6)}=\frac{Ax+B}{x^2+5}+\frac{C}{x-6}$$

- This process needs to be done for **each and every** factor.
- If we were to add these fractions together, we would end up with the same function on the left hand side.

Important Point – Be aware of factors that repeat

- If factors do repeat, you will need to make multiple fractions with denominators with increasing exponents. For example,
$$\frac{1}{x(x+2)^3}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{(x+2)^2}+\frac{D}{(x+2)^3}$$

  In general, for any factor that repeats (be it linear or quadratic), we will have

$$\frac{G(x)}{(factor)^i}=\frac{A_1}{factor}+\frac{A_2}{factor^2}+\ldots+\frac{A_{i-1}}{factor^{i-1}}+\frac{A_i}{factor^i}$$

- Repeating factors still take on their general form in the numerator. They are not raised to a power.
  - If linear, the numerator has the form of A
  - If quadratic, the numerator has the form of $Ax+B$

Step 3 – Multiply both sides of the equation by the common denominator.

- This removes the fractions from the equation.
- For our walkthrough example, we have

$$\frac{2x-6}{(x^2+5)(x-6)}=\frac{Ax+B}{x^2+5}+\frac{C}{x-6}$$

$$\left[\frac{2x-6}{(x^2+5)(x-6)}=\frac{Ax+B}{x^2+5}+\frac{C}{x-6}\right](x^2+5)(x-6)$$

$$2x-6=(Ax+B)(x-6)+C(x^2+5)$$

Step 4 – Solve for all constants using a system of linear equations

- Multiply everything together so that there are no factors and only individual terms. For example,

Copyright © 2007 Who Likes Homework - All rights reserved.

$$
\begin{aligned}
2x-6 &= (Ax+B)(x-6)+C(x^2+5) \\
&= Ax^2-6Ax+Bx-6B+Cx^2+5C
\end{aligned}
$$

- Now, group all of the terms with a common variable (i.e x), factor out the variable and reorder. For example

$$2x-6 = Ax^2+Cx^2-6Ax+Bx-6B+5C$$

$$2x-6 = (A+C)x^2+(-6A+B)x+(-6B+5C)$$

- Equate the coefficients between the left hand side and the right hand side.

$$2x-6 = (A+C)x^2+(-6A+B)x+(-6B+5C)$$

$$0x^2+2x-6 = (A+C)x^2+(-6A+B)x+(-6B+5C)$$

Thus, we have

$$
\begin{aligned}
0 &= A+C \\
2 &= -6A+B \\
-6 &= -6B+5C
\end{aligned}
$$

- Solve the system of linear equations
    - We find that C = 6/41 , A = -6/41 and B = 46/41

Step 5 – Plug the constants back into the original expression

$$\frac{2x-6}{x^3-6x^2+5x-30} = \frac{2x-6}{(x^2+5)(x-6)} = \frac{\left(-\frac{6}{41}\right)x+\left(\frac{46}{41}\right)}{x^2+5}+\frac{\left(\frac{6}{41}\right)}{x-6}$$

Step 6 – Integrate

- To finish off our example…

Copyright © 2007 Who Likes Homework - All rights reserved.

$$\int \frac{2x-6}{x^3-6x^2+5x-30}dx = \int \frac{\left(-6/41\right)x+\left(46/41\right)}{x^2+5}+\frac{\left(6/41\right)}{x-6}dx$$

$$= \int \frac{\left(-6/41\right)x+\left(46/41\right)}{x^2+5}dx + \int \frac{\left(6/41\right)}{x-6}dx$$

$$= -\frac{1}{41}\int \frac{6x-46}{x^2+5} + \frac{6}{41}\int \frac{1}{x-6}dx$$

$$= -\frac{1}{41}\left[\int \frac{6x}{x^2+5} - 46\int \frac{1}{x^2+5}\right] + \frac{6}{41}\int \frac{1}{x-6}dx$$

$$= -\frac{1}{41}\left[\int \frac{6x}{x^2+5} - 46\int \frac{1}{x^2+\left(\sqrt{5}\right)^2}\right] + \frac{6}{41}\int \frac{1}{x-6}dx$$

$$= -\frac{1}{41}\left(3\ln\left(x^2+5\right) - \frac{46}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}}\right) + \frac{6}{41}\ln\left(x-6\right) + C$$

<u>To recap the steps taken to solve rational fraction integrations</u>

Step 1 - Factor out the denominator as much as possible

Step 2 - Turn the function to a sum of fractions, whose denominators correspond to the factors determined in Step 1. The numerator will be an expression with unknown constants.

Step 3 - Multiply both sides of the equation by the common denominator to remove fractions.

Step 4 – Solve for all of the unknown constants

Step 5 – Plug the constants back into the sum of fractions from Step 2

Step 6 – Integrate

If you have any questions about this tutorial, or if you would like to see a tutorial on another topic, please contact Who Likes Homework at tutorials@wholikeshomework.com

Copyright © 2007 Who Likes Homework - All rights reserved.